Stat 821 Homework 6 Solution

Question 1

(a)

$$E\left(\frac{\partial}{\partial\theta}\log p(x,\theta)\right)=\frac{\partial}{\partial\theta}\log\frac{1}{\theta}=-\frac{1}{\theta}\neq 0$$

(b)

$$E\left(\frac{\partial}{\partial\theta}\log p(x,\theta)\right)^2=\frac{1}{\theta^2}$$

thus

$$Var\left(\frac{\partial}{\partial\theta}\log p(x,\theta)\right)=\frac{1}{\theta^2}-\left(-\frac{1}{\theta}\right)^2=0$$

Information bound is $+\infty$.

(c)

$$E(2X)=2E(X)=2\frac{\theta}{2}=\theta \qquad \text{unbiased}$$

$$Var(2X)=4Var(X)=\frac{\theta^2}{3}<\infty$$

Question 2 (Problem 5.16)

(a)

$$\begin{aligned}
I(\theta) &= E\left[\frac{\partial\log p(x,\theta)}{\partial\theta}\right]^2 \\
&= E\left[\frac{1}{\theta}\left(1+\frac{f'(x/\theta)}{f(x/\theta)}\frac{x}{\theta}\right)\right]^2 \\
&= \frac{1}{\theta^2}\int\left(1+\frac{f'(x/\theta)}{f(x/\theta)}\frac{x}{\theta}\right)^2\frac{1}{\theta}f(x/\theta)dx \\
&= \frac{1}{\theta^2}\int\left(1+\frac{f'(y)}{f(y)}y\right)^2 f(y)dy
\end{aligned}$$

(b)

$$I(\xi) = \frac{I(\theta)}{\left(\frac{d}{d\theta}\xi\right)^2} = \frac{I(\theta)}{(1/\theta)^2} = \int \left(1 + \frac{f'(y)}{f(y)}y\right)^2 f(y)dy$$

$I(\theta)$ is independent of θ.

Question 3 (Problem 6.5)

(a) Assume $X \sim N(\xi, \sigma^2)$,

$$l = \log p(x;\theta) = -\frac{1}{2}\log 2\pi\sigma^2 - \frac{(x-\xi)^2}{2\sigma^2}$$

$$\frac{\partial}{\partial\xi}l = \frac{(x-\xi)}{\sigma^2} \qquad \frac{\partial^2}{\partial\xi^2}l = -\frac{1}{\sigma^2} \qquad \frac{\partial}{\partial\sigma}l = -\frac{1}{\sigma} + \frac{(x-\xi)^2}{\sigma^3}$$

$$\frac{\partial^2}{\partial\xi\partial\sigma}l = -\frac{2(x-\xi)}{\sigma^3} \qquad \frac{\partial^2}{\partial\sigma^2}l = \frac{1}{\sigma^2} - \frac{3(x-\xi)^2}{\sigma^4}$$

Thus

$$I_{11} = -E(-1/\sigma^2) = \frac{1}{\sigma^2} \qquad I_{12} = I_{21} = -E\left(-\frac{2(x-\xi)}{\sigma^3}\right) = 0$$

$$I_{22} = -E\left(\frac{1}{\sigma^2} - \frac{3(x-\xi)^2}{\sigma^4}\right) = -\frac{1}{\sigma^2} + \frac{3\sigma^2}{\sigma^4} = \frac{2}{\sigma^2}$$

Assume $X \sim Gamma(\alpha, \beta)$,

$$l = \log p(x;\alpha,\beta) = -\log T(\alpha) - \alpha\log\beta - (\alpha-1)\log x - \frac{x}{\beta}$$

Let $\varphi(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$, then

$$\frac{\partial}{\partial\alpha}l = -\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \log\beta - \log x = -\varphi(\alpha) - \log\beta - \log x$$

$$\frac{\partial^2}{\partial\alpha^2}l = \frac{\Gamma''(\alpha)\Gamma(\alpha) - (\Gamma'(\alpha))^2}{(\Gamma(\alpha))^2} = -\varphi'(\alpha)$$

$$\frac{\partial^2}{\partial\alpha\partial\beta}l = -\frac{1}{\beta} \qquad \frac{\partial}{\partial\beta}l = -\frac{\alpha}{\beta} + \frac{x}{\beta^2} \qquad \frac{\partial^2}{\partial\beta^2}l = \frac{\alpha}{\beta^2} - \frac{2x}{\beta^3}$$

Thus

$$I_{11} = \varphi'(\alpha) \qquad I_{12} = I_{21} = \frac{1}{\beta} \qquad I_{22} = -E\left(\frac{\alpha}{\beta^2} - \frac{2x}{\beta^3}\right) = \frac{\alpha}{\beta^2}$$

Assume $X \sim B(\alpha, \beta)$,

$$\log p(x; \alpha, \beta) = \log\Gamma(\alpha+\beta) - \log\Gamma(\alpha) - \log\Gamma(\beta) + (\alpha-1)\log x + (\beta-1)\log(1-x)$$

$$\frac{\partial}{\partial\alpha} l = \frac{\Gamma'(\alpha+\beta)}{\Gamma(\alpha+\beta)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \log x = \varphi_\alpha(\alpha+\beta) - \varphi_\alpha(\alpha) + \log x$$

$$\frac{\partial^2}{\partial\alpha^2} l = \varphi'(\alpha+\beta) - \varphi'(\alpha)$$

$$\frac{\partial^2}{\partial\alpha\partial\beta} l = \varphi'(\alpha+\beta)$$

$$\frac{\partial}{\beta} l = \varphi_\beta(\alpha+\beta) - \varphi_\beta(\alpha) + \log(1-x)$$

$$\frac{\partial^2}{\partial\beta^2} l = \varphi'_\beta(\alpha+\beta) - \varphi'_\beta(\beta)$$

Thus

$$I_{11} = -\varphi'(\alpha+\beta) + \varphi'(\alpha) \qquad I_{12} = I_{21} = -\varphi'(\alpha+\beta)$$

$$I_{22} = -\varphi'_\beta(\alpha+\beta) + \varphi'_\beta(\beta)$$

(b) By chain rule

$$\frac{\partial}{\partial\xi_i} \log p_{\theta(\xi)}(x) = \sum_{k=1}^{s} \frac{\partial}{\partial\theta_k} \log p_{\theta(\xi)}(x) \frac{\partial\theta_k}{\partial\xi_i}$$

and

$$\frac{\partial}{\partial\xi_j} \log p_{\theta(\xi)}(x) = \sum_{l=1}^{s} \frac{\partial}{\partial\theta_l} \log p_{\theta(\xi)}(x) \frac{\partial\theta_l}{\partial\xi_j}$$

Therefore

$$\begin{aligned} & E\left[\frac{\partial}{\partial\xi_i} \log p_{\theta(\xi)}(X) \frac{\partial}{\partial\xi_j} \log p_{\theta(\xi)}(X)\right] \\ = \;& E\left[\sum_{k=1}^{s}\sum_{l=1}^{s} \left(\frac{\partial}{\partial\theta_k} \log p_{\theta(\xi)}(X)\right)\left(\frac{\partial}{\partial\theta_l} \log p_{\theta(\xi)}(X)\right)\left(\frac{\partial\theta_k}{\partial\xi_i}\right)\left(\frac{\partial\theta_l}{\partial\xi_j}\right)\right] \\ = \;& \sum_{k=1}^{s}\sum_{l=1}^{s} I_{kl}(\theta) \frac{\partial\theta_k}{\partial\xi_i}\frac{\partial\theta_l}{\partial\xi_j} \end{aligned}$$

The ij^{th} entry of matrix JIJ' is

$$\sum_{k=1}^{s}\sum_{l=1}^{s} I_{kl}(\theta) \frac{\partial\theta_k}{\partial\xi_i}\frac{\partial\theta_l}{\partial\xi_j} \qquad \text{where } \frac{\partial\theta_k}{\partial\xi_i} \text{ is the } ik^{th} \text{ element of J}$$

So $I^*(\xi) = JIJ'$.

Question 4 (Problem 1.3)

By Taylor series expansion,

$$h(c_n\bar{X}_n) = h(\xi)+h'(\xi)(c_n\bar{X}-\xi)+\frac{1}{2}h''(\xi)(c_n\bar{X}-\xi)^2+\frac{1}{6}h'''(\xi)(c_n\bar{X}-\xi)^3+o(c_nx_n,\xi)$$

$$\Rightarrow \quad E(h(c_n\bar{X}_n)) = h(\xi) + h'(\xi)\frac{a}{n}\xi + \frac{1}{2}h''(\xi)\frac{\sigma^2}{n} + o(1/n^2) \qquad (*)$$

$$\Rightarrow \quad E^2(h(c_n\bar{X}_n)) = h^2(\xi) + 2h(\xi)h'(\xi)\frac{a}{n}\xi + h(\xi)h''(\xi)\frac{\sigma^2}{n} + o(1/n^2)$$

$$(h^2(\xi))' = 2h(\xi)h'(\xi) \quad \text{and} \quad (h^2(\xi))'' = 2\{h(\xi)h''(\xi) + [h'(\xi)]^2\}$$

Check that h^2 satisfies the condition of Theorem1.1 also. So we have

$$E(h^2(c_n\bar{X}_n)) = h^2(\xi)+2h(\xi)h'(\xi)\frac{a}{n}\xi+(h(\xi)h''(\xi)+(h'(\xi))^2)\frac{\sigma^2}{n}+o(1/n^2)$$

by applying h^2 to $(*)$. Thus

$$Var[h^2(c_n\bar{X}_n)] = E(h^2(c_n\bar{X}_n)) - E^2(h(c_n\bar{X}_n)) = [h'(\xi)]^2\frac{\sigma^2}{n} + o(1/n^2)$$

Check

$$E(c_n\bar{X}_n - \xi)^2 = \frac{\sigma^2}{n} + o(1/n^2)$$

$$E(c_n\bar{X}_n-\xi)^i = \sum_{k=0}^{i}\binom{i}{k}c_n^kE(\bar{X}-\xi)^k(a/n+o(1/n^2))^{i-k}\xi^{i-k} = o(1/n^{i-1}) \quad \text{for } i \geq 3$$

Question 5 (Problem 1.33)

Let

$$Y_1, \ldots, Y_n \overset{iid}{\sim} Bernoulli(p)$$

$$X = \sum_{i=1}^{n} Y_i \qquad E(Y_i) = p \qquad Var(Y_i) = pq$$

Let

$$\delta' = \frac{x(n-x)}{n^2} = \bar{y}(1-\bar{y})$$

$$\delta = \frac{x(n-x)}{n(n-1)} = \bar{y}(1-\bar{y})\frac{n}{n-1}$$

$$h(t) = t(1-t) \quad h' = 1-2t \quad \text{if } t \neq 1/2, h' \neq 0$$

In theorem 1.10, set $c_n = 1, h(p) = pq$, then

$$\sqrt{n}(\delta' - pq) = \sqrt{n}(\bar{Y}(1-\bar{Y}) - pq) \quad \xrightarrow{D} \quad N(0, pq(1-2p)^2)$$

Thus

$$\sqrt{n}(\delta' - pq) = \sqrt{n}(\delta - \delta') + \sqrt{n}(\delta' - pq) = \frac{\sqrt{n}\bar{y}(1-\bar{y})}{n-1} + \sqrt{n}(\delta' - pq)$$

$$\text{Since } \bar{Y} \xrightarrow{P} p,\ 1-\bar{Y} \xrightarrow{P} q \Longrightarrow \bar{Y}(1-\bar{Y}) \xrightarrow{P} pq$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n-1} \to 0 \quad \text{so} \quad \frac{\sqrt{n}\bar{Y}(1-\bar{Y})}{n-1} \xrightarrow{P} pq$$

We have

$$\sqrt{n}(\delta - pq) \to N(0, pq(1-2p)^2) \quad \text{if } p \neq 1/2$$

If $p = 1/2$, i.e.

$$h'(p) = 1 - 2p = 0 \quad h''(p) = -2 \neq 0$$

then by theorem 1.10 again

$$n[\delta' - pq] \xrightarrow{D} \frac{1}{2}pq(-2)\chi_1^2 \sim -pq\chi_1^2$$

Similarly,

$$n[\delta - pq] = \frac{n}{n-1}\bar{y}(1-\bar{y}) + n(\delta' - pq)$$

$$\bar{Y}(1-\bar{Y}) \xrightarrow{P} pq, \qquad \frac{n}{n-1}\bar{Y}(1-\bar{Y}) \xrightarrow{P} pq = \frac{1}{4}$$

By Slustky theorem

$$n[\delta - pq] \xrightarrow{D} \frac{1}{4}(1 - \chi_1^2)$$